EQUITABLE COLORING OF SPARSE PLANAR GRAPHS

RONG LUO*, D. CHRISTOPHER STEPHENS†, AND GEXIN YU‡

Abstract. A proper vertex coloring of a graph G is equitable if the sizes of color classes differ by at most one. The equitable chromatic threshold $\chi^*_{eq}(G)$ of G is the smallest integer m such that G is equitably n-colorable for all $n \geq m$. We show that for planar graphs G with minimum degree at least two, $\chi^*_{eq}(G) \leq 4$ if the girth of G is at least 10, and $\chi^*_{eq}(G) \leq 3$ if the girth of G is at least 14.

1. Introduction

Graph coloring is a natural model for scheduling problems. Given a graph $G = (V, E)$, a proper vertex k-coloring is a mapping $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ if $uv \in E(G)$. The notion of equitable coloring is a model to equally distribute resources in a scheduling problem. A proper k-coloring f is equitable if

$$|V_1| \leq |V_2| \leq \ldots \leq |V_k| \leq |V_1| + 1,$$

where $V_i = f^{-1}(i)$.

The *equitable chromatic number* of G, denoted by $\chi_{eq}(G)$, is the smallest integer m such that G is equitably m-colorable. The *equitable chromatic threshold* of G, denoted by $\chi^*_{eq}(G)$, is the smallest integer m such that G is equitably n-colorable for all $n \geq m$. It is clear that $\chi_{eq}(G) \leq \chi^*_{eq}(G)$ for any graph G. They may be different: for example, $\chi_{eq}(K_{7,7}) = 2$ while $\chi^*_{eq}(K_{7,7}) = 8$.

Hajnal and Szemerédi [2] proved that $\chi^*_{eq}(G) \leq \Delta(G) + 1$ for any graph G with maximum degree $\Delta(G)$. The following conjecture made by Chen, Lih and Wu [1], if true, strengthens the above result.

Conjecture 1 (Chen, Lih and Wu [1])**.** *For any connected graph G different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$, $\chi^*_{eq}(G) \leq \Delta(G)$.*

Except for some special cases, the conjecture is still wide open in general.

Another direction of research on equitable coloring is to consider special families of graphs.

For planar graphs, Zhang and Yap [5] proved that a planar graph is equitably m-colorable if $m \geq \Delta \geq 13$. When the girth $g(G)$ is large, fewer colors are needed.

Theorem 1.1 (Wu and Wang, [4])**.** *Let G be a planar graph with $\delta(G) \geq 2$.*
(a) If $g(G) \geq 26$, then $\chi^*_{eq}(G) \leq 3$;
(b) If $g(G) \geq 14$, then $\chi^*_{eq}(G) \leq 4$.

The purpose of this paper is to improve the above two results. Our main results are contained in the following theorems.

Theorem 1.2. *If G is a planar graph with $\delta(G) \geq 2$ and $g(G) \geq 10$, then $\chi^*_{eq}(G) \leq 4$.*

Date: March 6, 2009.
Key words and phrases. equitable coloring, planar graphs, girth.
* Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, TN 37132 rluo@mtsu.edu. Research supported in part by MTSU summer and academic year research grant 2008-2009.
† Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, TN 37132 cstephen@mtsu.edu. Research supported in part by NSA grant H98230-07-1-0006.
‡ Department of Mathematics, College of William and Mary, Williamsburg, VA, 23185 gyu@wm.edu. Research supported in part by the NSF grant DMS-0852452.

Theorem 1.3. *If G is a planar graph with $\delta(G) \geq 2$ and $g(G) \geq 14$, then $\chi^*_{eq}(G) \leq 3$.*

Since $K_{1,n}$ is not equitably k-colorable when $n \geq 2k-1$, we cannot drop the requirement of $\delta(G) \geq 2$ in the theorems. On the other hand, we do not believe that the girth conditions are best possible. It would be very interesting to find the best possible girth condition for both 3- and 4-equitable coloring.

2. Preliminaries

Before starting, we introduce some notation. A *k-vertex* is a vertex of degree k; a $(\geq k)$- and a $(\leq k)$*-vertex* have degree at least and at most k, respectively. A *thread* is a path with 2-vertices in its interior and (≥ 3)-vertices as its endpoints. A k-thread has k interior 2-vertices. If a (≥ 3)-vertex u is the endpoint of a thread containing a 2-vertex v, and the distance between u and v on the thread is $l+1$, then we say that u and v are *loosely l-adjacent*. Thus "loosely 0-adjacent" is the same as the usual "adjacent."

All of our proofs rely on the techniques of reducibility and discharging. We start with a minimal counterexample G to the theorem we are proving, and the idea of the reduction is as follows. We remove a small subgraph H (for instance, a vertex of degree at least three, together with its incident 2-threads) from the graph G. We have an equitable k-coloring f of $G-H$, and we attempt to extend f to an equitable coloring of G. This can be done if we can equitably k-color H itself, with some extra conditions: namely, the color classes which should be "large" in H are predetermined by the existing coloring of $G-H$; and secondly, the parts of H with edges to $G-H$ have color restrictions. Such a graph H is called a *reducible configuration.*

Let the *maximum average degree* of G be $mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$. A planar graph G with girth at least g has maximum average degree $mad(G) < \frac{2g}{g-2}$. We let the initial charge at vertex v be $M(v) = d(v) - \frac{2g}{g-2}$. We will introduce some rules to re-distribute the charges (discharging), and after the discharging process, every vertex v has a final charge $M'(v)$. Note that

$$\sum_{v\in V(G)} M'(v) = \sum_{v\in V(G)} M(v) = \sum_{v\in V(G)} (d(v) - \frac{2g}{g-2}) < 0. \tag{1}$$

We will show that either we have some reducible configurations, or the final charges are all non-negative. The former contradicts the assumption that G is a counterexample, and the latter contradicts (1).

We will prove the theorems on 3-coloring and 4-coloring separately. Before the proofs, we provide some properties useful to equitable m-coloring with $m \geq 3$.

Let $m \geq 3$ be an integer. Let G be a graph which is not equitably m-colorable with $|V|+|E|$ as small as possible.

Observation 2.1. *G is connected.*

Proof. Let $H_1, H_2, \cdots, H_k$ be the connected components of G where $k \geq 2$. By the choice of G, both $H = H_1 \cup H_2 \cup H_{k-1}$ and H_k are equitably m-colorable. An equitable m-coloring of H with $|V_1(H)| \geq |V_2(H)| \geq \cdots \geq |V_m(H)|$ and an equitable m-coloring of H_k with $|V_1(H_k)| \leq |V_2(H_k)| \leq \cdots \leq |V_m(H_k)|$ induce an equitable m-coloring of G, contradicting the choice of G. □

3. Equitable 4-coloring

In this section, we prove Theorem 1.2. We start with some useful lemmas.

The following lemma is an extension of a fact first observed in [3], and we use it to prove Lemma 3.2.

Lemma 3.1. *Let $S = \{x_1, x_2, ..., x_m\}$ be a set of m distinct vertices in G. Suppose each x_i, $i = 2, \ldots, m$, has the same m colors available, and x_1 has at least one color available. If $G-S$ has an equitable m-coloring, and*

$$|N_G(x_i) - S| \leq m - i \tag{2}$$

for $1 \le i \le m$, then G has an equitable m-coloring.

Lemma 3.2. *Let G be a graph and $P = y_0y_1...y_ty_{t+1}$ such that $t \in \{4,5\}$, and $d(y_i) = 2$ for each $i = 1,..,t$. Let $m \ge 4$ be an integer and $a, b \in \{1, 2, ..., m\}$. Let x be an arbitrary vertex in $\{y_1, y_2, ..., y_t\}$. If $G - \{y_1, ..., y_t\}$ has an equitable m-coloring f, then f can be extended to G such that $f(x) \ne a, b$ unless $m = 4$, $t = 5$, and $x = y_4$ or $x = y_2$.*

Proof. Let $V_1, \cdots, V_m$ be the m color classes of $G - S$ under f with $|V_1| \le |V_2| \le \cdots \le |V_m|$, where $S = \{y_1, y_2, \cdots, y_t\}$. Assume $x = y_i$ for some $i \le \lceil \frac{t}{2} \rceil$ by symmetry.

When $m \ge t$, we arrange the y_i's into a list $x_1, x_2, \ldots, x_t$ such that $x_1 = x$ and $|N(x_t) \cap S| = 2$, and assign every vertex other than x with the same color list $\{1, 2, \ldots, t\}$, and assign x a color different from a, b and from the color of the neighbor of x in $G - S$. Then by Lemma 3.1, we can extend f to G such that $f(x) \notin \{a, b\}$.

If $m < t$, then $m = 4$ and $t = 5$, and in this case, $x \in \{y_1, y_3\}$. If $1 \notin \{a, b, f(y_0)\}$, then assign 1 to y_1 and y_3, assign a color $c_4 \in \{2, 3, 4\} - \{f(y_6)\}$ to y_5, and assign the other two colors in $\{2, 3, 4\} - \{c_4\}$ arbitrarily to y_2 and y_4. If $1 \in \{f(y_0), a, b)\}$, then $|\{2, 3, 4\} - \{f(y_0), a, b\}| \ge 1$. Let $\{x, x'\} = \{y_1, y_3\}$ and $c_2 \in \{2, 3, 4\} - \{f(y_0), a, b\}$. Assign 1 to y_2 and y_4 and c_2 to x; assign a color $c_3 \in \{2, 3, 4\} - \{c_2, f(y_0)\}$ to x'; and assign the remaining color $c_4 \in \{2, 3, 4\} - \{c_2, c_3\}$ to y_5. If $c_4 = f(y_6)$, swap colors on y_5 and y_4. In either case, f can be extended to G. □

Lemma 3.3. *Let xy_1y_2y be a 2-thread of G and $m \ge 4$ be an integer. If G has an m-equitable coloring f such that $f(x) \ne f(y)$, then f can be extended to G.*

Proof. Let f be an equitable m-coloring of $G - \{y_1, y_2\}$ and let $V_1, \cdots, V_m$ be the m color classes with $|V_1| \le |V_2| \le \cdots \le |V_m|$. If $f(x) \ne f(y)$, then there is a bijection $\phi : \{1, 2\} \mapsto \{1, 2\}$ such that $\phi(1) \ne f(x)$ and $\phi(2) \ne f(y)$. Assign $\phi(1)$ to y_1 and $\phi(2)$ to y_2. Hence f can be extended to G. □

Lemma 3.4. *Let xy_1y_2y be a 2-thread and xy_3z be a 1-thread incident with x. Let $m \ge 4$ be an integer. If $G - \{y_1, y_2, y_3\}$ has an equitable m-coloring f with $f(x) \notin \{f(y), f(z)\}$, then f can be extended to G.*

Proof. Let $V_1, \cdots, V_m$ be the m color classes with $|V_1| \le |V_2| \le \cdots \le |V_m|$. If $f(x) \in \{1, 2, 3\}$, let $a = f(x)$ and then $a \ne f(y)$. Otherwise choose a color $a \in \{1, 2, 3\}$ such that $a \ne f(y)$. Let $b \in \{1, 2, 3\} - \{a, f(z)\}$ and $c \in \{1, 2, 3\} - \{a, b\}$. Then $b \notin \{f(x), f(z)\}$ and $c \ne f(x)$. Assign a to y_2, b to y_3 and c to y_1. Thus we obtain an equitable m-coloring of G. □

Proof of Theorem 1.2 Let G be a minimal counterexample to Theorem 1.2 with $|V| + |E|$ as small as possible. That is, G is planar with $\delta(G) \ge 2$ and girth at least 10, and G is not equitably m-colorable for some integer $m \ge 4$ but any proper subgraph of G with minimum degree at least 2 is equitably m-colorable for each $m \ge 4$.

Claim 3.1. *G has no t-thread with $t \ge 3$, and G has no thread with same endvertices.*

Proof. Let $P = v_0v_1 \cdots v_tv_{t+1}$ be a t-thread in G with $t \ge 3$.

If $v_0 \ne v_{t+1}$ or $d(v_0) \ge 4$, consider $G_1 = G - \{v_1, \cdots, v_t\}$. Then $\delta(G_1) \ge 2$. By the choice of G, G_1 has an equitable m-coloring. Let $V_1, \cdots, V_m$ be the m color classes with $|V_1| \le |V_2| \le \cdots \le |V_m|$. We can extend the coloring to G to obtain an equitable m-coloring of G as follows: first color the vertex v_i by the color k where $k \equiv i \pmod m$ for each $i = 1, \cdots t$. Swap colors of v_1 and v_2 if the colors of v_1, v_0 are the same, and further swap the colors of v_{t-1}, v_t if there is any conflict between v_t and v_{t+1}.

Now assume that $v_0 = v_{t+1}$ and $d(v_0) = 3$. Let $x \in N(v_0)$ and $x \ne v_1, v_t$. If $d(x) \ge 3$, consider $G_2 = G - \{v_0, v_1, \cdots, v_t\}$. Then $\delta(G_2) \ge 2$. By the choice of G, G_2 has an equitable m-coloring. Let $V_1, \cdots, V_m$ be the m color classes with $|V_1| \le |V_2| \le \cdots \le |V_m|$. We can extend the coloring to G to obtain an equitable m-coloring of G as follows: first color the vertex v_i the color k where $k \equiv i \pmod m$; if $0 \equiv t \pmod m$, swap the colors of v_t and v_{t-1}; if the colors of x, v_0 are the same, further swap the colors of v_0, v_i where $1 \le i \le 2$ and the color of v_i is different from that of v_t (such v_i exists since v_0, v_1, v_2 are colored differently).

If $d(x) = 2$, then let $Q = x_0x_1 \cdots x_qx_{q+1}$ be the thread containing the edge v_0x_1 where $x_1 = x$ and $x_0 = v_0$. Consider the graph $G_3 = G - \{v_0, x_1, \cdots, x_q, v_1, \cdots, v_t\}$. Then $\delta(G_3) \geq 2$. By the choice of G, G_3 has an equitable m-coloring. Let $V_1, \cdots, V_m$ be the m color classes with $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. We first extend the coloring to G to obtain an equitable m-coloring of $G - \{v_1, \cdots, v_t\}$ as follows: first color the vertex x_i the color k where $k \equiv i + 1 (\text{mod } m)$ for each $i = 0, 1, \cdots q$; if x_q and x_{q+1} have the same color, swap the colors of x_q and x_{q-1}. Then we further extend the coloring to G similarly to the case where $d(v_0) \geq 4$. □

Let x be a vertex of degree $d = d(x) \geq 3$. Then x is the endvertex of d threads. Denote $T(x) = (a_2, a_1, a_0)$ where a_i is the number of i-threads incident with x. Denote $t(x) = 2a_2 + a_1$.

Claim 3.2. *For a 4-vertex x, $t(x) \leq 5$.*

Proof. Suppose $t(x) \geq 6$. By Claim 3.1, x is not incident with any t-thread where $t \geq 3$. Since $t(x) \geq 6$, x is incident with at least two 2-threads. Label two 2-threads incident with x as $xx_1z_1y_1$, $xx_2z_2y_2$.

We first show that x is incident with at most two 2-threads. Suppose that x is incident with at least three 2-threads. Label the third 2-thread incident with x as: $xx_3z_3y_3$. Label the fourth thread incident with x as $xx_4z_4y_4$, xz_4y_4, or xy_4, depending on whether it is a 2-thread, or a 1-thread, or a 0-thread. Denote $A = \{x, x_i, z_i | 1 \leq i \leq 4\}$, $A = \{x, x_j, z_i | 1 \leq i \leq 4, 1 \leq j \leq 3\}$, or $A = \{x, x_i, z_i | 1 \leq i \leq 3\}$, depending on whether x is incident with a 0-thread, a 1-thread, or four 2-threads, respectively. By the choice of G, $G - A$ has an equitable m-coloring f.

Now if x is not incident with a 0-thread, then by Lemma 3.2, f can be extended to $G - \{x_1, x_2, z_1, z_2\}$ such that $f(x) \notin \{f(y_1), f(y_2\}$. By Lemma 3.3, it can be further extended to $G - \{x_1, z_1\}$ since $f(x) \neq f(y_2)$ and to G since $f(x) \neq f(y_1)$. This contradicts the choice of G.

If, on the other hand, x is incident with a 0-thread, then first extend the coloring f of $G - A$ to $G - \{x_1, z_1\} - xy_4$ such that $f(x) \notin \{f(y_1), f(y_4)\}$ by Lemma 3.2. Since $f(x) \neq f(y_4)$, it is also an equitable m-coloring of $G - \{x_1, z_1\}$. Since $f(x) \neq f(y_1)$, by Lemma 3.3, the coloring of $G - \{x_1, z_1\}$ can be extended to G, a contradiction. This proves that x is incident with at most two 2-threads.

Now we have $t(x) \leq 6$. Since $t(x) \geq 6$ and x is incident with at most two 2-threads, we have $t(x) = 6$ and $T(x) = (2, 2, 0)$. Label the two 1-threads incident with x as xx_3y_3 and xx_4y_4. Then $G - \{x, z_1, z_2, x_i | 1 \leq i \leq 4\}$ has an equitable m-coloring. Since $y_3x_3xx_2z_2y_2$ is a 4-thread in $G - \{x_1, z_1, x_4\}$, by Lemma 3.2, f can be extended to $G - \{x_1, z_1, x_4\}$ such that $f(x) \notin \{f(y_1), f(y_4)\}$. By Lemma 3.4, it can be further extended to G since $f(x) \neq f(y_1)$. This contradicts the choice of G, proving Claim 3.2. □

Claim 3.3. *For a 3-vertex x, either $t(x) \leq 2$ or $T(x) = (1, 2, 0)$ and $m = 4$.*

Proof. We first prove that $T(x) \neq (1, 2, 0)$ if $m \geq 5$. Suppose $T(x) = (1, 2, 0)$ and $m \geq 5$. Label the two 1-thread incident with x as xx_1y_1 and xx_2y_2 and label the 2-thread as $xx_3x_4y_3$. Let $A = \{x, x_1, x_2, x_3, x_4\}$. Then $\delta(G - A) \geq 2$ and it has an equitable m-coloring f. Let $V_1, \cdots, V_m$ be the m color classes with $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. Let $\{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}$ such that $a \neq f(y_1)$, $b \neq f(y_2)$, and $c \neq f(x_4)$. Assign a to x_1, b to x_2, c to x_4, d to x, and e to x_3. It is easy to see that the extension of f is an equitable m-coloring of G, contradicting the choice of G.

Now assume $T(x) \neq (1, 2, 0)$. We prove $t(x) \leq 2$. Suppose $t(x) \geq 3$ and $T(x) \neq (1, 2, 0)$. By Claim 3.1, x is not incident with any t-thread where $t \geq 3$. We first consider the case where x is not incident with a 2-thread. Then $T(x) = (0, 3, 0)$. Label the three 1-threads incident with x as xx_iy_i for $i = 1, 2, 3$. Note that $d(y_i) \geq 3$ and $d(x_i) = 2$. Consider the graph $G_1 = G - \{x, x_1, x_2, x_3\}$. Then $\delta(G_1) \geq 2$ and by the choice of G, G_1 has an equitable m-coloring. Let $V_1, \cdots, V_m$ be the m color classes with $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. Let $\{1, 2, 3, 4\} = \{a, b, c, d\}$ such that no color in $\{a, b, c\}$ is used by all three vertices y_1, y_2, y_3. An equitable m-coloring of G can be obtained by coloring the vertices x_1, x_2, x_3 with the colors a, b, c such that no conflict occurs and coloring the vertex x with the color d. This contradicts the choice of G. Hence $T(x) \neq (0, 3, 0)$ and x is incident with at least one 2-thread.

Now we consider the case $a_2 \neq 0$. Let xx_1x_2y be one 2-thread incident with x. If $t(x) \geq 5$, then $G - \{x_1, x_2\}$ has minimum degree 2 and has a t-thread, say P, containing x for some $4 \leq t \leq 5$. Let G_1 be the subgraph obtained from $G - \{x_1, x_2\}$ by further deleting the degree-two vertices in P. Then G_1 has an equitable m-coloring f. By Lemma 3.2, f can be extended to $G - \{x_1, x_2\}$ such that $f(x) \neq f(y)$. By Claim 3.3, f can be further extended to G. This contradicts the choice of G. Assume $3 \leq t(x) \leq 4$. Since x is incident with at least one 2-thread and $T(x) \neq (1, 2, 0)$, x must be incident with a 0-thread. Call it xu. Since $t(x) \geq 3$, $G - xu$ has a t-thread P containing x where $4 \leq t \leq 5$. Let G_2 be the subgraph obtained from $G - xu$ by further deleting the degree-two vertices in P. Then G_2 has an equitable m-coloring f. By Lemma 3.2, f can be extended to $G - xu$ such that $f(x) \neq f(u)$. The extension of f is also an equitable m-coloring of G, a contradiction.

This completes the proof of Claim 3.3. □

A 3-vertex x is *bad* if $T(x) = (1, 2, 0)$. Note that if $m \geq 5$, the configuration $T(x) = (1, 2, 0)$ with $d(x) = 3$ is still reducible; thus there are no bad 3-vertices when $m \geq 5$. The following two claims deal with two reducible configurations for $m = 4$.

Claim 3.4. *Assume $m = 4$. Let x be a bad 3-vertex and y be a vertex loosely 1-adjacent to x. Then*
(1) if $d(y) = 3$, then $t(y) = 1$;
(2) if $d(y) = 4$, then y is loosely 1-adjacent to at most one bad 3-vertex.

Proof. Label the threads incident with x as: $xx_1x_2u_1$, xx_3u_2, and xx_4y.

(1) Suppose that $d(y) = 3$ and $t(y) \geq 2$. Then by Claim 3.4, either $t(y) = 2$ or y is a bad 3-vertex. If $t(y) = 2$, then $T(y) = (0, 2, 0)$. If y is a bad 3-vertex, then $T(y) = (1, 2, 0)$. In either case, y is incident with exactly two 1-threads. Label the other 1-thread incident with y as: yy_1z. Label the third thread incident with y as yu or yy_2y_3u depending on whether it is a 0-thread or a 2-thread. Denote $A = \{x, y, y_1, x_i | 1 \leq i \leq 4\}$; denote $B = \emptyset$ if y is incident with a 0-thread, and $B = \{y_2, y_3\}$ otherwise. Consider the graph $G_1 = G - [A \cup B]$. Then $\delta(G_1) \geq 2$ and by the choice of G, G_1 has an equitable 4-coloring. Any 4-equitable coloring of G_1 can be extended to $G - A$. Hence $G - A$ has an equitable 4-coloring f. By Lemma 3.2, f can be extended to $G - \{x_1, x_2, x_3\}$ such that $f(x) \notin \{f(u_1), f(u_2)\}$. (If $f(y) = f(u)$ in case of $B = \emptyset$ or $f(y) = f(y_2)$ in case of $B \neq \emptyset$, switch colors on y and x_4.) By Lemma 3.4, the equitable 4-coloring of $G - \{x_1, x_2, x_3\}$ can be further extended to G. This contradicts the choice of G, proving (1).

(2) Suppose that $d(y) = 4$ and y is loosely 1-adjacent to two bad 3-vertices, say x, z. Label the threads incident with z as $zz_1z_2u_3$, zz_3u_4, and zy_1y. Let u_5 and u_6 be the endvertices of the two threads incident with y other than the ones adjacent to x and z. Denote $A = \{x, y, z, y_1, x_i, z_j | 1 \leq i \leq 4, 1 \leq j \leq 3\}$. Let B denote the set of 2-vertices on the two threads incident with y other than yy_1z and yx_4x. Let $G_1 = G - [A \cup B]$. Then $\delta(G_1) \geq 2$ and G_1 has an equitable 4-coloring f. Let $|V_1| \leq |V_2| \leq |V_3| \leq |V_4|$ be the 4 color classes. Note that if y is incident with a 2-thread, then f can be extended to the 2-vertices in the 2-thread. In the following, without loss of generality, we assume that y is not incident with a 2-thread.

We first consider the case where y is incident with exactly two 1-threads: xx_4y and zy_1y. Then $B = \emptyset$. By Lemma 3.2, extend f to y_1yx_4x such that $f(y) \notin \{f(u_5), f(u_6)\}$. Note that the colors $f(x), f(x_4), f(y), f(y_1)$ are different. If $f(x) \in \{f(u_1), f(u_2)\}$, then either $f(x_4)$ or $f(y_1)$ is not in $\{f(u_1), f(u_2)\}$. If $f(x_4) \notin \{f(u_1), f(u_2)\}$, switch the colors on $f(x)$ and $f(x_4)$. If $f(y_1) \notin \{f(u_1), f(u_2)\}$, switch the colors on $f(x)$ and $f(y_1)$. Hence we have an extension of f on xx_4yy_1 such that $f(x) \notin \{f(u_1), f(u_2)\}$. By Lemmas 3.2 and 3.4, f can be further extended to G, a contradiction.

Now we consider the case where y is incident with at least three 1-threads. Label the third 1-thread as yy_2u_5 and the fourth thread incident with y as yy_3u_6 or yu_6 depending on whether it is a 0-thread or a 1-thread. Then either $B = \{y_2\}$ or $B = \{y_2, y_3\}$ depending on whether the fourth thread incident with y is a 0-thread or a 1-thread. We first extend f to $\{x_4, y, y_1\} \cup B$.

Let a, b be two colors in $\{1,2,3,4\} - \{f(u_5), f(u_6)\}$. If $B = \{y_2\}$, then assign a, b randomly to y_2 and y. Assign the remaining two colors randomly to x_4 and y_1. If $B = \{y_2, y_3\}$, then assign a, b randomly to y_2 and y_3 first. If $1 \notin \{a, b\}$, assign 1 to x_4 and y_1 and the fourth color to y. If $1 \in \{a, b\}$, assign 1 to x_4 and the remaining two colors to y and y_1 randomly. By Lemma 3.2, the equitable 4-coloring can be further extedned to the two 4-paths $x_2x_1xx_3$ and $z_2z_1zz_3$. Thus we may obtain an equitable 4-coloring of G. It contradict the choice of G, proving (2). □

Since $g(G) \geq 10$, we have $mad(G) < 2.5$. Let $M(x) = d(x) - 2.5$ be the *initial charge* of x for $x \in V$. We will re-distrubte the charges among vertices according to the *discharging rules* below:

(R1) Each 2-vertex receives $\frac{1}{4}$ from each of the endvertices of the thread containing it.

(R2) If x is a bad 3-vertex, x receives $\frac{1}{4}$ from each loosely 1-adjacent vertex.

Let $M'(x)$ be the charge of x after application of rules R1 and R2. The following Claim shows a contradiction to (1), which implies the truth of Theorem 1.2.

Claim 3.5. *$M'(x) \geq 0$ for each $x \in V$.*

Proof. If $d(x) = 2$, then $M'(x) = 2 - 2.5 + \frac{2}{4} = 0$.

Assume $d(x) = 3$. If x is not a bad vertex, then by Claim 3.4, $t(x) \leq 2$ and x sends out at most $2 \times \frac{1}{4} = \frac{1}{2}$. If x is a bad vertex, then $t(x) = 4$ and it sends out $2 \times \frac{1}{4} = 1$. It also receives $\frac{1}{4}$ from each loosely 1-adjacent vertex. Hence $M'(x) \geq 3 - 2.5 - 1 + 2 \times \frac{1}{4} = 0$.

Assume $d(x) = 4$. Then x is loosely 1-adjacent to at most one bad 3-vertex by Claim 3.4. Hence x sends out at most $\max\{\frac{t(x)}{4}, \frac{1}{2} + \frac{t(x)-1}{4}\} = \frac{t(x)+1}{4} \leq \frac{3}{2}$ since $t(x) \leq 5$ by Claim 3.2. Therefore $M'(x) \geq 4 - 2.5 - \frac{3}{2} = 0$.

Assume $d(x) \geq 5$. Let y be a (≥ 3)-vertex that is loosely k-adjacent to x. If $k = 2$, then x sends out $2 \times \frac{1}{4}$ via this 2-thread. If $k = 1$, then x sends out $\frac{1}{4}$ via this thread if y is not a bad vertex and sends out $2 \times \frac{1}{4} = \frac{1}{2}$ via this 1-thread if y is a bad vertex. In summary, x sends out at most $\frac{1}{2}$ via each thread incident with it. Hence $M'(x) \geq d(x) - 2.5 - \frac{d(x)}{2} = \frac{d(x)}{2} - 2.5 \geq 0$. □

4. Reduction lemmas for equitable 3-coloring

We now proceed to equitable 3-coloring. We first prove two lemmas which give conditions for the existence of reducible configurations.

As we described in Section 2, a subgraph H is reducible if we can equitably 3-color H, such that the color classes which should be "large" in H are predetermined by the existing coloring of $G - H$, and the parts of H with edges to $G - H$ have color restrictions.

We will handle the latter condition by means of lists of allowed colors in H. We will handle the former condition by predetermining the sizes of the color classes. Thus we have the following definition.

Definition 2. *Let H be a graph with list assignment $L = \{l_v\}$, with $l_v \subset \{1, 2, \ldots, k\}$. Call H* descending-equitably L-colorable *if H can be L-colored such that $|V_1| \geq |V_2| \geq \cdots \geq |V_k| \geq |V_1| - 1$.*

Note that if $G - H$ has an equitable k-coloring with $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$, then G is equitably k-colorable if H is descending-equitably L-colorable. Because of this, a descending-equitably L-colorable subgraph H is a reducible configuration in G.

Let a_i^v be the number of i-threads incident to vertex v. If it is clear from the context, we drop v in the notation. The following two lemmas give simple ways to identify reducible configurations using relations involving a_i^v.

Lemma 4.1 (Reducing a vertex with at most one 0-thread)**.** *Let S be a subdivided star of order s with root x, and let $L = \{l_v\}$ be a list assignment to the vertices of S such that $l_v = \{1, 2, 3\}$ if v is not a leaf or the root, $l_v \subset \{1, 2, 3\}$ with $|l_v| = 2$ if v is a leaf, and $|l_x| \geq 2$. Let $d(x) \leq 6$ and assume $a_i = 0$ unless $i \in \{0, 1, 2, 4\}$. If $2a_4 + a_2 \geq a_1 + 1 + \varepsilon$ and $a_4 \geq d(x) - 4$, then S is descending-equitably L-colorable, where $\varepsilon = 3\lceil s/3 \rceil - s$.*

Proof. Let c, c' be the colors allowed at x. Let p_i $(1 = 1, 2, 3)$ be the desired size of V_i. Let S_i $(i = c, c')$ be a maximum independent set such that S_i contains the root, and such that $i \in l_v$ for all $v \in S_i$. Then no vertex of a 1-thread is in any of the S_i; each 2-thread contains a vertex (the leaf) that is in at least one of S_i's; and for each 4-thread, the leaf is in at least one of the S_i's, and the vertex 1-vertex-away from the root is in both of the S_i's. Thus $|S_c|+|S_{c'}| \geq 2+3a_4+a_2$.

If $|S_c| + |S_{c'}| > 2(\lceil s/3\rceil - 1)$, then some color may be assigned to the root and extended to a large enough independent set. Thus we assume that $2 + 3a_4 + a_2 \leq |S_c| + |S_{c'}| \leq 2(\lceil s/3\rceil - 1) = \frac{2}{3}(s+\epsilon-3) = \frac{2}{3}(4a_4+2a_2+a_1+1)$. Therefore we have $a_4+12 \leq a_2+2(a_1+\epsilon+1) \leq a_2+2(2a_4+a_2)$, that is $a_4 + a_2 \geq 4$. So $a_2 + 2(a_1 + \epsilon + 1) \geq a_4 + 2 \geq 6 - a_2$, and we have $a_1 + a_2 + \epsilon \geq 7$, a contradiction since $a_1 + a_2 \leq 4$ and $\epsilon \leq 2$.

Let c be the color assigned to the root, and let c' and c'' be the other two colors. Without loss of generality assume we want $|V_{c'}| \geq |V_{c''}|$. We color with c' first. By hypothesis, we know that $2a_4+a_2 \geq \lceil \frac{s}{3} \rceil$, so there is an independent set I large enough for c' which misses the root x, all leaves, and the vertices colored with c. Color I with c', color the remaining vertices with c''. Now I contains no leaves, and it is easy to see we can pair the vertices colored c'' with vertices colored c' such that if there is any conflict on a leaf, the colors can be switched without altering the rest of the coloring. □

Lemma 4.2 (Reducing two vertices connected by a 1-thread; one vertex may have one 0-thread)**.** *Suppose x and y are connected by a 1-thread, and $d(x) + d(y) \leq 8$. Let S be the graph of order s induced by the union of the subdivided star with root x and the subdivided star with root y. Let $L = \{l_v\}$ be a list assignment to the vertices of S such that $l_v = \{1, 2, 3\}$ if v is not a leaf and $v \neq y$, and $l_v \subset \{1, 2, 3\}$ with $|l_v| = 2$ if v is a leaf or $v = y$. Let $b_i = a_i^x + a_i^y$ for $i = 0, 2, 4$, and let $b_1 = a_1^x + a_1^y - 1$. Then S is descending-equitably L-colorable if $2b_4 + b_2 \geq b_1 - 1 + \varepsilon$ and $b_4 \geq 1$, where $\varepsilon = 3\lceil s/3\rceil - s$.*

Proof. Let c, c' be the colors allowed at y. Let p_i $(1 = 1, 2, 3)$ be the desired size of V_i. Note that $p_c + p_{c'} \leq \lfloor s/3\rfloor + \lceil s/3\rceil$. Let S_i $(i = c, c')$ be a maximum independent set such that S_i contains x and y, and such that $i \in l_v$ for all $v \in S_i$. Then no vertex of a 1-thread is in any of the S_i; each 2-thread contains a vertex (the leaf) that is in at least one of S_i's; and for each 4-thread, the leaf is in at least one of the S_i's, and the vertex 1-vertex-away from the root is in both of the S_i's. Thus $|S_c| + |S_{c'}| \geq 4 + 3b_4 + b_2$.

If $|S_c| + |S_{c'}| > 2(\lceil s/3\rceil - 1)$, then some color may be assigned to the root and extended to a large enough independent set. Thus we assume that $4 + 3b_4 + b_2 \leq |S_c| + |S_{c'}| \leq 2(\lceil s/3\rceil - 1) = \frac{2}{3}(s+\epsilon-3) = \frac{2}{3}(4b_4+2b_2+b_1+2)$. Therefore we have $b_4+14 \leq b_2+2(b_1+\epsilon) \leq b_2+2(2b_4+b_2)+2$, that is $b_4+b_2 \geq 4$. So $b_2+2(b_1+\epsilon) \geq b_4+14 \geq 18-b_2$, and we have $b_1+b_2+\epsilon \geq 9$, a contradiction since $b_1 + b_2 \leq 6$ and $\epsilon \leq 2$.

Let c be the color assigned to x and y, and let c' and c'' be the other two colors. Without loss of generality assume we want $|V_{c'}| \geq |V_{c''}|$. We color with c' first. By hypothesis, we know that $2b_4 + b_2 \geq \lceil \frac{s}{3} \rceil$, so there is an independent set I large enough for c' which misses x and y, which misses all leaves, and which moreover misses the vertices colored with c. Color I with c', color the remaining vertices with c''. Now I contains no leaves, and it is easy to see we can pair the vertices colored c'' with vertices colored c' such that if there is any conflict on a leaf, the colors can be switched without altering the rest of the coloring. □

5. Equitable 3-coloring

In this section, we prove Theorem 1.3.

By Theorem 1.2, we only need to show that planar graphs with minimum degree at least two and girth at least 14 are equitably 3-colorable. Suppose not, and let G be a counterexample with $|V| + |E|$ as small as possible.

Claim 5.1. *G has no t-thread where $t = 3$ or $t \geq 5$, and no thread with the same endpoints.*

Proof. The proof is more or less a line by line copy of the proof of Claim 3.1, so we omit it here. □

Similarly to Section 3, for a vertex x, let $T(x) = (a_4, a_2, a_1, a_0)$, where a_i is the number of i-threads incident to x, and let $t(x) = 4a_4 + 2a_2 + a_1$.

Claim 5.2. *Let x be a vertex with $3 \le d(x) \le 6$. Then*
(a) if $d(x) = 3$, then either $t(x) \le 4$ or $T(x) = (1, 0, 2, 0)$;
(b) if $d(x) = 4$, then $t(x) \le 7$ or $T(x) = (2, 0, 0, 2)$;
(c) if $d(x) \in \{5, 6\}$, then $a_4 \le d(x) - 2$.

Proof. Assume that $t(x) \ge 5$ when $d(x) = 3$, $t(x) \ge 8$ when $d(x) = 4$, and $a_4 \ge d(x) - 1$ when $d(x) \in \{5, 6\}$.

Note that when $d(x) \in \{3, 5, 6\}$, $a_0 \le 1$. When $d(x) = 4$, $a_0 > 1$ and $t(x) \ge 8$ only if $a_4 = a_0 = 2$, in which case we are done. So we may assume $a_0 \le 1$, thus Lemma 4.1 applies.

Let H be the graph formed by x and its loosely adjacent 2-vertices. Then $G - H$ has an equitable 3-coloring f, and we may assume that f cannot be extended to H. Thus by Lemma 4.1,

$$2a_4 + a_2 \le a_1 + \epsilon, \tag{3}$$

where $\epsilon = 3\lceil \frac{|V(H)|}{3} \rceil - |V(H)|$. Since $t(x) = 4a_4 + 2a_2 + a_1$, we have

$$t(x) = 2(2a_4 + a_2) + a_1 \le 3a_1 + 2\epsilon. \tag{4}$$

Let $d(x) = 3$. By (4), $a_1 \ge 1$. Then $(a_4, a_2) \in \{(1, 1), (2, 0), (1, 0), (0, 2)\}$ If $(a_4, a_2) = (1, 1)$, then $\epsilon = 1$ and $a_1 = 1$, a contradiction to (3); if $(a_4, a_2) = (2, 0)$, then $a_1 = 1$ and $\epsilon = 2$, a contradiction to (3) again; and if $(a_4, a_2) = (0, 2)$, then $a_1 = 1$ and $\epsilon = 0$, another contradiction to (3). So $(a_4, a_2) = (1, 0)$. It follows that $a_1 = 2$ or $a_1 = a_0 = 1$. If $a_1 = a_0 = 1$, then $\epsilon = 0$, a contradiction to (3). Therefore $a_1 = 2$ and $T(x) = (1, 0, 2, 0)$.

Let $d(x) = 4$. By (4), $a_1 \ge 2$. Then $(a_4, a_2) \in \{(1, 1), (2, 0)\}$. If $(a_4, a_2) = (1, 1)$, then $\epsilon = 0$, and by (3), $a_1 \ge 3$ and thus $a_4 + a_2 \le 1$, a contradiction; if $(a_4, a_2) = (2, 0)$, then $\epsilon = 1$ and thus $a_1 = 2$, a contradiction to (3).

If $d(x) \in \{5, 6\}$, then $a_1 \le 1$, and clearly we have a contradiction to (3). □

We call a 3-vertex x *bad* if $T(x) = (1, 0, 2, 0)$

Claim 5.3. *Let x be a bad 3-vertex. Let y be a 3-vertex which is loosely 1-adjacent to x. Then*
(a) y is not in a t-thread where $t \ge 2$; hence $t(y) \le 3$; and
(b) x is the only bad 3-vertex to which y is loosely 1-adjacent.

Proof. (a) Suppose y is a 3-vertex which is loosely 1-adjacent to x. Suppose also that y is incident with a $t-$thread where $t \ge 2$. Let H be the union of x, y, and all 2-vertices loosely adjacent to x or y. We apply Lemma 4.2 to H, observing that $b_4 = a_4^y + 1$, $b_2 = a_2^y$, and $b_1 = a_1^y + 1$. We find that H is reducible if $2b_4 + b_2 \ge b_1 + \varepsilon - 1$ or equivalently if $2a_4^y + a_2^y \ge a_1^y + \epsilon - 2.$.

Now if $a_1^y = 1$, then we are done because $2a_4^y + a_2^y \ge 1 \ge \epsilon - 1 = a_1^y + \epsilon - 2$. Since y is adjacent to a t-thread with $t \ge 2$, it must be that $a_1^y = 2$. Thus we may reduce H if $2a_4^y + a_2^y \ge \varepsilon$. This is obviously true if $a_4^y > 0$. Thus we may assume $a_2^y = 1$. But in this case $|H| = 11$, $\varepsilon = 1$, and $a_2^y \ge \varepsilon$. Thus H is reducible.

(b) Suppose now that y is a 3-vertex which is loosely 1-adjacent to x and another bad 3-vertex, z. Let H be the graph induced by x, y, z, and all the 2-vertices loosely adjacent to x, y, or z. Let G' be $G - H$. G' is equitably 3-colorable by induction, and we need to extend this equitable 3-coloring to all of G. We will 3-color H, and for $i = 1, 2, 3$ let U_i be the set of vertices of H colored by i. For the coloring to remain equitable, we need $|U_1| \ge |U_2| \ge |U_3| \ge |U_1| - 1$. Call a proper coloring of H "good" if it satisfies $|U_1| \ge |U_2| \ge |U_3| \ge |U_1| - 1$.

The union of x, y, z together with the 1-threads at x and z forms a 9-path; let us label it as $v_1 w_1 x w_2 y w_3 z w_4 v_2$. Label the 4-thread at x as $x x_1 x_2 x_3 x_4 v_3$, and label the 4-thread at z as $z z_1 z_2 z_3 z_4 v_4$.

First suppose that y is adjacent to a 0-thread. Then $|U_i|$ should be 5 for all i, and some color is disallowed at y by its adjacency to G'. Assume without loss of generality that 3 is an allowed color at y. Let $U'_1 = \{w_1, w_4, x_1, x_3, z_3\}$, $U'_2 = \{w_2, w_3, x_4, z_1, z_4\}$, and $U'_3 = \{x, y, z, x_2, z_2\}$. This is a good coloring of H, so it only remains to repair any conflicts at the leaves of H when H

is attached to G'. Notice that if there is a conflict with the leaf adjacent to w_1, we may simply switch the colors on w_1 and w_2. Likewise we may pair w_3 with w_4, x_3 with x_4, and z_3 with z_4, switching any pair if there is a conflict at the associated leaf. Any such switch results in another good coloring of H, and switching any pair does not interfere with any other pair. Thus we may obtain appropriate U_i in this case.

If y is incident to a third 1-thread with 2-vertex y_1, then we keep the U_i's as before and color y_1 by 1. Note that y_1 and z_1 form another switchable pair if there is a conflict at y_1.

By (a), y is not incident to any t-thread with $t \geq 2$, so the proof of the claim is complete. □

Since $g(G) \geq 14$, we have $mad(G) < \frac{7}{3}$. Let $M(x) = d(x) - \frac{7}{3}$ be the *initial charge* of x for $x \in V$. We will re-distribute the charges among vertices according to the *discharging rules* below:

(R1) Every (≥ 3)-vertex sends $\frac{1}{6}$ to each loosely adjacent 2-vertex;

(R2) Every (≥ 3)-vertex sends $\frac{1}{6}$ to each loosely 1-adjacent bad 3-vertex.

Let $M'(x)$ be the final charge of x. The following Claim shows a contradiction to (1), which in turn implies the truth of Theorem 1.3.

Claim 5.4. *For each $x \in V$, $M'(x) \geq 0$.*

Proof. If $d(x) = 2$, then $M'(x) = 2 - \frac{7}{3} + 2 \cdot \frac{1}{6} = 0$.

If $d(x) = 3$, then if x is bad, it gains two $\frac{1}{6}$ from each of the two loosely 1-adjacent vertices, thus $M'(x) = 3 - \frac{7}{3} - 6 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} = 0$; if x is not bad and is not loosely 1-adjacent to to a bad vertex, then $M'(x) = 3 - \frac{7}{3} - 4 \cdot \frac{1}{6} = 0$; if x is not bad and is loosely 1-adjacent to a bad 3-vertex, then $t(x) \leq 3$, thus $M'(x) = 3 - \frac{7}{3} - 3 \cdot \frac{1}{6} - \frac{1}{6} = 0$.

For $d(x) \geq 4$, note that $M'(x) = d(x) - \frac{7}{3} - \frac{(4a_4+2a_2+2a_1)}{6}$. Since $d(x) = a_4 + a_2 + a_1 + a_0$, we have

$$M'(x) = \frac{1}{3}(2d(x) - 7 - a_4 + a_0).$$

When $d(x) \geq 7$, $M'(x) \geq (d(x) - a_4 + a_0)/3 \geq 0$. When $d(x) \in \{5, 6\}$, by Claim 5.2 $a_4 \leq d(x) - 2$, thus $M'(x) \geq 0$.

Now consider x with $d(x) = 4$. To show $M'(x) \geq 0$, it suffices to show that $a_4 \leq a_0 + 1$, which is true, since by Claim 5.2, $a_4 \leq 1$. □

References

[1] B. Chen, K. Lih, and P. Wu, Equitable coloring and the maximum degree, *European J. Combin.* **15** (1994) 443-447.

[2] A. Hajnal, E. Szemerédi, Proof of a conjecture of Erdős, in: P. Erdős, A. Rényi, V.T. Sós (Eds.), Combinatorial Theory and its applications, vol. II, Colloquia Mathematica Societatis János Bolyai, vol. 4, North-Holland, Amsterdam, 1970, pp. 601-623.

[3] A. Kostochka, M. Pelsmajor, and D. West, A List Analogue of Equitable Coloring, *J. Graph Theory*, **44** (2003) 166-177.

[4] J. Wu and P. Wang, Equitable coloring planar graphs with large girth, *Discrete Mathematics*, **308** (2008) 985–990.

[5] Y. Zhang and H. Yap, Equitable colorings of planar graphs, *J. Combin. Math. Combin. Comput.* **27** (1998) 97–105.